Modeling and Solving the Balanced Assignment Problem with CP Optimizer

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1 The problem

This problem is described in dmcommunity.org/challenge/challenge-sep-2018. It deals with assignment of people to certain activities, for instance: assigning students to project groups or assigning professors to offices. Let’s assume we have 210 people that belong to 4 categories as described in the file BalancedAssignment.csv:

<table>
<thead>
<tr>
<th>NAME</th>
<th>DEPARTMENT</th>
<th>LOCATION</th>
<th>GENDER</th>
<th>TITLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>BIW</td>
<td>NNE</td>
<td>Peoria</td>
<td>M</td>
<td>Assistant</td>
</tr>
<tr>
<td>KRS</td>
<td>WSW</td>
<td>Springfield</td>
<td>F</td>
<td>Assistant</td>
</tr>
<tr>
<td>TLR</td>
<td>NNW</td>
<td>Peoria</td>
<td>F</td>
<td>Adjunct</td>
</tr>
<tr>
<td>VAA</td>
<td>NNW</td>
<td>Peoria</td>
<td>M</td>
<td>Deputy</td>
</tr>
<tr>
<td>JRT</td>
<td>NNE</td>
<td>Springfield</td>
<td>M</td>
<td>Deputy</td>
</tr>
</tbody>
</table>

You need to assign these 210 people to 12 groups of 16, 17, 18, or 19 people in every group. Of course, a person may be assigned to only one group. The objective is to maximize the diversity of people in the groups. Diversity means that it is preferable to assign people with the same characteristics to different groups. In this note we use a definition of diversity that is using a simple rule that measures the penalty when two people with similar characteristics are assigned to the same group and then minimize the total penalty. More precisely, the penalty for having two people with similar characteristics in the same group is equal to the number of characteristics they have in common. For instance if BIW and JRT are in the same group, it will incur a penalty of 2 because they have two characteristics in common: their department (NNE) and their gender (M). Thus the problem is to assign the people to the different groups while minimizing the total penalty.

2 Formal definition

Let \( N \) denote the set of people and \( C \) the set of categories (here: DEPARTEMENT, LOCATION, GENDER, TITLE). For a person \( i \in N \) and a characteristic \( c \in C \), the value of this characteristic is denoted \( t_{ik} \). The penalty for having two persons \( i \) and \( j \) in the same group is: \( P_{ij} = |\{k \in C : t_{ik} = t_{jk}\}| \). Let \( G \) denote the set of groups, \( A \) (resp. \( B \)) the minimal (resp. maximal) number of people in a group. The problem can be defined as follows:
\[
\min \sum_{i,j \in N, i < j} P_{ij} \cdot (x_i == x_j)
\]
\[
A \leq \text{count}([x_i]_{i \in N}, g) \leq B \quad \forall g \in G
\]
\[
\text{integer } x_i \in G \quad \forall i \in N
\]

Where \(\text{count}([x_i]_{i \in N}, g)\) returns the number of variables \(x_i\) for \(i \in N\) such that \(x_i = g\) that is, the number of people in group \(g\). Note that for a pair of persons \(\{i, j\}\) in the same group, we apply the penalty only once (and not for both \((i, j)\) and \((j, i)\) which would result in a total penalty twice as big).

### 3 Complexity

The Balanced Assignment Problem is **NP-Complete** in the strong sense. The decision version of the problem is clearly in NP. We show NP-Completeness by a transformation from graph K-colorability.

Let \(G = (V, E)\) and \(K \leq |V|\) be an instance of graph K-colorability problem with \(V = \{1, ..., n\}\). We can transform this instance into an instance of Balanced Assignment Problem with \(K\) groups (one for each color) where the problem is to decide whether there exists a solution of total penalty 0. Each vertex \(i \in V\) will be a person. For each edge \((i, j) \in E\), we define a category \(C_{ij}\) as follow:

\[
C_{ij}(i) = -1
\]
\[
C_{ij}(j) = -1
\]
\[
C_{ij}(k) = k \quad \forall k \in V, k \neq i, k \neq j
\]

As the total penalty needs to be 0, the penalty associated with each of the \(K\) groups needs to be 0. It means that two persons \(i, j\) such that \((i, j) \in E\) cannot belong to the same group (same color) as otherwise, the penalty due to having them sharing the same characteristic in the same group \((C_{ij} = -1)\) would be 1. So any solution to the Balanced Assignment Problem is a valid K-coloring of the graph.

Reciprocally, any solution to the K-coloring problem will be such that for each pair \((k, l)\) of elements in a group (same color), and for every category \(C_{ij}\), we necessarily have \(\{i, j\} \neq \{k, l\}\) and thus \(C_{ij}(k) \neq C_{ij}(l)\). This means that the total penalty will be 0 and the solution is also a solution to the associated Balanced Assignment Problem.

Though the problem is NP-Complete, the above transformation from graph K-colorability requires a lot of categories in the corresponding Balanced Assignment Problem (at most \(n^2\) as we create one category per edge in the graph). In practice, I think that when the problem uses few categories (4 in the example), then the problem is not very difficult and very good solution can be build using simple heuristics suggested by the computation of lower bounds in section 6.

### 4 A CP Optimizer formulation

The complete CP Optimizer formulation of the problem in Python is given on Fig. \[\text{Fig. 1}\]. For more information on CP Optimizer, we refer interested readers to [1].

### 5 A solution

Using this model we can find a solution with total penalty \(3370\) shown on Fig. \[\text{Fig. 2}\]. We will see in the next section that this solution is **optimal** because we show, using some simple mathematical considerations, that a lower bound on the total penalty is 3370.
The distribution of people characteristics in this solution is shown on the plots on Fig. 3 and 4.

6 Computing a lower bound

Let $S$ denote the set of people characteristics (for instance a characteristic can be \texttt{DEPARTMENT=NNE} or \texttt{GENDER=M}). We denote $h_{is} = 1$ if person $i \in N$ has characteristic $s \in S$, $h_{is} = 0$ otherwise. Given two persons $i$ and $j$ the penalty for having them in the same group is the number of characteristics they have in common, that is:

$$P_{ij} = \sum_{s \in S} h_{is} \cdot h_{js}$$

Suppose we have a feasible solution. For a group $g \in G$, we denote $E_g$ the set of persons in group $g$. The total penalty of the solution is:
Figure 2: Details of an optimal solution

<table>
<thead>
<tr>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>G4</th>
<th>G5</th>
<th>G6</th>
<th>G7</th>
<th>G8</th>
<th>G9</th>
<th>G10</th>
<th>G11</th>
<th>G12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>DHY</td>
<td>TLR</td>
<td>MJR</td>
<td>MES</td>
<td>VAA</td>
<td>KRS</td>
<td>B1W</td>
<td>DCN</td>
<td>CWT</td>
<td>DBN</td>
</tr>
<tr>
<td>2</td>
<td>SWR</td>
<td>CWU</td>
<td>LLR</td>
<td>HCN</td>
<td>JRS</td>
<td>LVO</td>
<td>AMR</td>
<td>JRT</td>
<td>ECA</td>
<td>A1H</td>
<td>REN</td>
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<tr>
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<td>OGZ</td>
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<td>TFG</td>
<td>EAZ</td>
<td>DWO</td>
<td>CAE</td>
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<td>WMK</td>
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<tr>
<td>4</td>
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<td>CAH</td>
<td>CGH</td>
<td>MAI</td>
<td>SJO</td>
<td>JDS</td>
<td>DHE</td>
<td>CPR</td>
<td>PWS</td>
<td>PMO</td>
<td>WMS</td>
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<tr>
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<td>K1Y</td>
<td>TEN</td>
<td>JWS</td>
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<td>DGR</td>
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<td>JPY</td>
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<td>J1R</td>
<td>PAS</td>
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<td>C1N</td>
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<td>ET1</td>
<td>D1L</td>
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<td>PSA</td>
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<td>ASO</td>
<td>EHR</td>
</tr>
<tr>
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<td>RE1</td>
<td>TBR</td>
<td>S1R</td>
<td>T1</td>
<td>AJU</td>
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<td>RGE</td>
<td>SER</td>
<td>K1N</td>
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<tr>
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<td>L1S</td>
<td>BLW</td>
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<tr>
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<td>SCG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\sum_{g \in G} \sum_{i,j \in E_g, i < j} P_{ij} = \sum_{g \in G} \sum_{i,j \in E_g, i < j} \sum_{s \in S} h_{is} \cdot h_{js}
\]

\[
= \sum_{s \in S} \sum_{g \in G} \sum_{i,j \in E_g, i < j} h_{is} \cdot h_{js}
\]

\[
= \sum_{s \in S} \frac{1}{2} \cdot \sum_{i \in E_g} h_{is} \cdot (\sum_{i \in E_g} h_{is} - 1)
\]

\[
= \frac{1}{2} \sum_{s \in S} \sum_{g \in G} n_{gs}(n_{gs} - 1)
\]

Where \(n_{gs} = \sum_{i \in E_g} h_{is}\) denotes the number of persons with characteristic \(s\) allocated to group \(g\).

We see that the objective function is separable by characteristics \(s \in S\). To compute a lower bound, we will consider \(|S|\) independent problems with only the people with characteristic \(s\). These \(N_s = \sum_{i \in E_g} h_{is}\) persons have to be partitioned into \(|G|\) groups so as to minimize \(\sum_{g \in G} n_{gs}(n_{gs} - 1)\). If we relax the integrality on \(n_{gs}\), a trivial optimal solution is to equilibrate the number of people in each group, so taking \(n_{gs} = N_s/|G|\). Thus a valid lower bound for the original problem is:

\[
\frac{1}{2} \cdot \sum_{s \in S} \frac{N_s}{|G|} \cdot (\frac{N_s}{|G|} - 1)
\]

In fact it is easy to show that an optimal solution for characteristic \(s\) is to have \(|G| \cdot (\frac{N_s}{|G|} + 1) - N_s\) groups with \(\frac{N_s}{|G|}\) persons and \(N_s - |G| \cdot \frac{N_s}{|G|}\) groups with \(\frac{N_s}{|G|} + 1\) persons. This gives a total penalty for characteristic \(s\) equal to:

\[
\frac{N_s}{|G|} \cdot (2 \cdot N_s - |G| \cdot (\frac{N_s}{|G|} - 1))
\]
Figure 3: Distribution of *department* and *location* among groups

Distribution of employees in each group by department

Distribution of employees in each group by location
Figure 4: Distribution of *gender* and *title* among groups

Distribution of employees in each group by gender

Distribution of employees in each group by title

- Consultant
- Assistant
- Adjunct
- Deputy
Thus, a valid lower bound for the original problem is:

\[
\sum_{s \in S} \left\lfloor \frac{N_s}{|G|} \right\rfloor \cdot (2 \cdot N_s - |G| \cdot \left(\frac{N_s}{|G|} - 1\right))
\]

If we compute it on the problem instance it gives a lower bound of 3370.

References